Some new characterizations of PST-groups

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Abstract

Let H and B be subgroups of a finite group G such that $G = N_G(H)B$. Then we say that H is *quasipermutable* (respectively S-*quasipermutable*) in G provided H permutes with B and with every subgroup (respectively with every Sylow subgroup) A of B such that $(|H|, |A|) = 1$. In this paper we analyze the influence of S-quasipermutable and quasipermutable subgroups on the structure of G. As an application, we give new characterizations of soluble PST-groups.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover p is always supposed to be a prime and π is a subset of the set $\mathbb{P}$ of all primes; $\pi(G)$ denotes the set of all primes dividing $|G|$.

A subgroup H of G is said to be *quasinormal* or *permutable* in G if H permutes with every subgroup A of G, that is, $HA = AH$; H is said to be S-*permutable* in G if H permutes with every Sylow subgroup of G.

A group G is called a PT-*group* if permutability is a transitive relation on G, that is, every permutable subgroup of a permutable subgroup of G is permutable in G. A group G is called a PST-*group* if S-permutability is a transitive relation on G.

As well as T-groups, PT-groups and PST-groups possess many interesting properties (see Chapter 2 in [1]). The general description of PT-groups and PST-groups were firstly obtained by Zacher [2] and Agrawal [3], for the soluble case, and by Robinson in [4], for the general case. Nevertheless,

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in the further publications, the authors (see for example recent papers[5]– [16]) have found out and described many other interesting characterizations of soluble PT and PST-groups.

In this paper we give new "Hall"-characterizations of soluble PST-groups on the basis of the following

Definition 1.1. We say that a subgroup H is *quasipermutable* (respectively S-*quasipermutable*) in G provided H permutes with B and with every subgroup (respectively with every Sylow subgroup) A of B such that $(|H|, |A|) = 1$.

Examples and some applications of quasipermutable subgroups were discussed in our papers [17] and [18] (see also remarks in Section 5 below). In this paper, we give the following result, which we consider as one more motivation for introducing the concept of quasipermutability.

Theorem A. *Let $D = G^{\mathfrak{N}}$ and $\pi = \pi(D)$. Then the following statements are equivalent:*

(i) *D is a Hall subgroup of G and every Hall subgroup of G is quasipermutable in G.*

(ii) *G is a soluble PST-group.*

(iii) *Every subgroup of G is quasipermutable in G.*

(iv) *Every π-subgroup of G and some minimal supplement of D in G are quasipermutable in G.*

In the proof Theorem A we use the next three our results.

A subgroup S of G is called a *Gaschütz* subgroup of G (L.A. Shemetkov [19, IV, 15.3]) if S is supersoluble and for any subgroups $K \leq H$ of G, where $S \leq K$, the number $|H : K|$ is not prime.

Theorem B. *The following statements are equivalent:*

(I) *G is soluble, and if S is a Gaschütz subgroup of G, then every Hall subgroup H of G satisfying $\pi(H) \subseteq \pi(S)$ is quasipermutable in G.*

(II) *G is supersoluble and the following hold:*

(a) *$G = DC$, where $D = G^{\mathfrak{N}}$ is an abelian complemented subgroup of G and C is a Carter subgroup of G;*

(b) *$D \cap C$ is normal in G and $(p, |D/D \cap C|) = 1$ for all prime divisors p of $|G|$ satisfying $(p - 1, |G|) = 1$.*

(c) *For any non-empty set π of primes, every π-element of any Carter subgroup of G induces a power automorphism on the Hall π'-subgroup of D.*

(III) *Every Hall subgroup of G is quasipermutable in G.*

Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be a *formation* if either $\mathfrak{F} = \varnothing$ or $1 \in \mathfrak{F}$ and every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for any group G. The formation $\mathfrak{F}$ is said to be *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$. A subgroup H of G is said to be an $\mathfrak{F}$-*projector* of G provided $H \in \mathfrak{F}$ and $E = E^{\mathfrak{F}}H$ for any subgroup E of G containing H. By the Gaschütz's theorem

[20, VI, 9.5.4 and 9.5.6], for any saturated formation $\mathfrak{F}$, every soluble group G has an $\mathfrak{F}$-projector and any two $\mathfrak{F}$-projectors of G are conjugate.

Theorem C. *Let $\mathfrak{F}$ be a saturated formation containing all nilpotent groups. Suppose that G is soluble and let $\pi = \pi(C) \cap \pi(G^{\mathfrak{F}})$, where C is an $\mathfrak{F}$-projector of G. If every maximal subgroup of every Sylow p-subgroup of G is S-quasipermutable in G for all $p \in \pi$, then $G^{\mathfrak{F}}$ is a Hall subgroup of G.*

Theorem D. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and $\pi = \pi(F^*(G^{\mathfrak{F}}))$. If $G^{\mathfrak{F}} \neq 1$, then for some $p \in \pi$ some maximal subgroup of a Sylow p-subgroup of G is not S-quasipermutable in G.*

In this theorem $F^*(G^{\mathfrak{F}})$ denotes the generalized Fitting subgroup of $G^{\mathfrak{F}}$, that is, the product of all normal quasinilpotent subgroups of $G^{\mathfrak{F}}$.

The main tool in the proofs of Theorems C and D is the following our result.

Proposition. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E such that $|P| > p$.*

(i) *If every number V of some fixed $\mathcal{M}_\phi(P)$ is S-quasipermutable in G, then E is p-supersoluble.*

(ii) *If every maximal subgroup of P is S-quasipermutable in G, then every chief factor of G between E and $O_{p'}(E)$ is cyclic.*

(iii) *If every maximal subgroup of every Sylow subgroup of E is S-quasipermutable in G, then every chief factor of G below E is cyclic.*

In this proposition we write $\mathcal{M}_\phi(G)$, by analogy with [21], to denote a set of maximal subgroups of G such that $\Phi(G)$ coincides with the intersection of all subgroups in $\mathcal{M}_\phi(G)$.

Note that Proposition may be independently interesting because this result unifies and generalize many known results, and in particular, Theorems 1.1–1.5 in [21] (see Section 5). In Section 5 we discus also some further applications of the results.

All unexplained notation and terminology are standard. The reader is referred to [19], [22], or [23] if necessary.

2 Basic Propositions

Let H be a subgroup of G. Then we say, following [17], that H is *propermutable* (respectively *S-propermutable*) in G provided there is a subgroup B of G such that $G = N_G(H)B$ and H permutes with all subgroups (respectively with all Sylow subgroups) of B.

Proposition 2.1. *Let $H \leq G$ and N a normal subgroup of G. Suppose that H is quasipermutable (S-quasipermutable) in G.*

(1) *If either H is a Hall subgroup of G or for every prime p dividing $|H|$ and for every Sy-*

low p-subgroup H_p of H we have $H_p \nleq N$, then HN/N is quasipermutable (S-quasipermutable, respectively) in G/N.

(2) *If $\pi = \pi(H)$ and G is π-soluble, then H permutes with some Hall π'-subgroup of G.*

(3) *H permutes with some Sylow p-subgroup of G for every prime p dividing $|G|$ such that $(p, |H|) = 1$.*

(4) *$|G : N_G(H \cap N)|$ is a π-number, where $\pi = \pi(N) \cup \pi(H)$.*

(5) *If H is propermutable (S-propermutable) in G, then HN/N is propermutable (S-propermutable, respectively) in G/N.*

(6) *If H is S-propermutable in G, then H permutes with some Sylow p-subgroup of G for any prime p dividing $|G|$.*

(7) *Suppose that G is π-soluble. If H is a Hall π-subgroup of G, then H is propermutable (S-propermutable, respectively) in G.*

Proof. By hypothesis, there is a subgroup B of G such that $G = N_G(H)B$ and H permutes with B and with all subgroups (with all Sylow subgroups, respectively) A of B such that $(|H|, |A|) = 1$.

(1) It is clear that

$$G/N = (N_G(H)N/N)(BN/N) = N_{G/N}(HN/N)(BN/N).$$

Let K/N be any subgroup (any Sylow subgroup, respectively) of BN/N such that $(|HN/N|, |K/N|) = 1$. Then $K = (K \cap B)N$. Let B_0 be a minimal supplement of $K \cap B \cap N$ to $K \cap B$. Then $K/N = (K \cap B)N/N = B_0(K \cap B \cap N)N/N = B_0N/N$ and $K \cap B \cap N \cap B_0 = N \cap B_0 \leq \Phi(B_0)$. Therefore $\pi(K/N) = \pi(K \cap B/K \cap B \cap N) = \pi(B_0)$, so $(|HN/N|, |B_0|) = 1$. Suppose that some prime $p \in \pi(B_0)$ divides $|H|$, and let H_p be a Sylow p-subgroup of H. We shall show that $H_p \nleq N$. In fact, we may suppose that H is a Hall subgroup of G. But in this case, H_p is a Sylow p-subgroup of G. Therefore, since $p \in \pi(B_0) \subseteq \pi(G/N)$, $H_p \nleq N$. Hence p divides $|HN/N|$, a contradiction. Thus $(|H|, |B_0|) = 1$, so in the case, when H is quasipermutable in G, we have $HB_0 = B_0H$ and hence HN/N permutes with $K/N = B_0N/N$. Thus HN/N is quasipermutable in G/N.

Finally, suppose that H is S-quasipermutable in N. In this case, B_0 is a p-subgroup of B, so for some Sylow p-subgroup B_p of B we have $B_0 \leq B_p$ and $(|H|, p) = 1$. Hence $K/N = B_0N/N \leq B_pN/N$, which implies that $K/N = B_pN/N$. But H permutes with B_p by hypothesis, so HN/N permutes with K/N. Therefore HN/N is S-quasipermutable in G/N.

(2) By [20, VI, 4.6], there are Hall π'-subgroups E_1, E_2 and E of $N_G(H)$, B and G, respectively, such that $E = E_1E_2$. Then H permutes with all Sylow subgroups of E_2 by hypothesis, so

$$HE = H(E_1E_2) = (HE_1)E_2 = (E_1H)E_2 =$$

$$E_1(HE_2) = E_1(E_2H) = (E_1E_2)H = EH$$

by [22, A, 1.6].

(3) See the proof of (2).

(4) Let p be a prime such that $p \notin \pi$. Then by (3), there is a Sylow p-subgroup P of G such that $HP = PH$ is a subgroup of G. Hence $HP \cap N = H \cap N$ is a normal subgroup of HP. Thus p does not divide $|G : N_G(H \cap N)|$.

(5) See the proof of (1).

(6) See the proof of (2).

(7) Since G is π-soluble, B is π-soluble. Hence by [20, VI, 1.7], $B = B_\pi B_{\pi'}$ where B_π is a Hall π-subgroup of B and $B_{\pi'}$ is a Hall π'-subgroup of B. By [20, VI, 4.6], there are Hall π-subgroups N_π, B_π and G_π of $N_G(H)$, B and G, respectively, such that $G_\pi = N_\pi B_\pi$. But since $H \leq N_\pi$, N_π is a Hall π-subgroup of G. Therefore $G_\pi = N_\pi B_\pi = N_\pi$, so $B_\pi \leq N_\pi$. Hence $G = N_G(H)B = N_G(H)B_\pi B_{\pi'} = N_G(H)B_{\pi'}$, so H is propermutable (S-propermutble, respectively) in G.

A group G is said to be a *C_π-group* provided G has a Hall π-subgroup and any two Hall π-subgroups of G are conjugate.

On the basis of Proposition 2.1 the following two results are proved.

Proposition 2.2. *Let H be a Hall S-quasipermutable subgroup of G. If $\pi = \pi(|G : H|)$, then G is a C_π-group.*

Proposition 2.3. *Let E be a normal subgroup of G and H a Hall π-subgroup of E. If H is nilpotent and S-quasipermutable in G, then E is π-soluble.*

3 Groups with a Hall quasipermutable subgroup

A group G is said to be *π-separable* if every chief factor of G is either a π-group or a π'-group. Every π-separable group G has a series

$$1 = P_0(G) \leq M_0(G) < P_1(G) < M_1(G) < \ldots < P_t(G) \leq M_t(G) = G$$

such that

$$M_i(G)/P_i(G) = O_{\pi'}(G/P_i(G))$$

$(i = 0, 1, \ldots, t)$ and

$$P_{i+1}(G)/M_i(G) = O_\pi(G/M_i(G))$$

$(i = 1, \ldots, t)$

The number t is called the *π-length* of G and denoted by $l_\pi(G)$ (see [34, p. 249]).

One more result, which we use use in the proof of our main results, is the following

Theorem 3.1. *Let H be a Hall subgroup of G and $\pi = \pi(H)$. Suppose that H is quasipermutable in G.*

(I) *If $p > q$ for all primes p and q such that $p \in \pi$ and q divides $|G : N_G(H)|$, then H is normal in G.*

(II) *If H is supersoluble, then G is π-soluble.*

(III) *If H is π-separable, then the following fold:*

(i) *$H' \leq O_\pi(G)$. If, in addition, $N_G(H)$ is nilpotent, then $G' \cap H \leq O_\pi(G)$.*

(ii) *$l_\pi(G) \leq 2$ and $l_{\pi'}(G) \leq 2$.*

(iii) *If for some prime $p \in \pi'$ a Hall π'-subgroup E of G is p-supersoluble, then G is p-supersoluble.*

Let $\mathcal{M}$ and $\mathcal{H}$ be non-empty formations. Then the *product* $\mathcal{MH}$ of these formations is the class of all groups G such that $G^{\mathcal{H}} \in \mathcal{M}$. It is well-known that such an operation on the set of all non-empty formations is associative (Gaschütz). The symbol $\mathcal{M}^t$ denotes the product of t copies of $\mathcal{M}$.

We shall need following well-known Gaschütz-Shemetkov's theorem [26, Corollary 7.13].

Lemma 3.2. *The product of any two non-empty saturated formations is also a saturated formation.*

In in the proof of Theorem 3.1 we use the following

Lemma 3.3. *The class $\mathcal{F}$ of all π-separable groups G with $l_\pi(G) \leq t$ is a saturated formation.*

Proof. It is not difficult to show that for any non-empty set $\omega \subseteq \mathbb{P}$ the class $\mathcal{G}_\omega$ of all ω-groups is a saturated formation and that $\mathcal{F} = (\mathcal{G}_{\pi'}\mathcal{G}_\pi)^t\mathcal{G}_{\pi'}$. Hence $\mathcal{F}$ is a saturated formation by Lemma 3.2.

Lemma 3.4. *Suppose that G is separable. If Hall π-subgroups of G are abelian, then $l_\pi(G) \leq 1$.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. Let N be a minimal normal subgroup of G. Since G is π-separable, N is a π-group or a π'-group. It is clear that the hypothesis holds for G/N, so $l_\pi(G/N) \leq 1$ by the choice of G. By Lemma 3.3, the class of all π-soluble groups with $l_\pi(G) \leq 1$ is a saturated formation. Therefore N is a unique minimal normal subgroup of G, $N \nleq \Phi(G)$ and N is not a π'-group. Hence N is a π-group and $N = C_G(N)$ by [22, A, 15.2]. Therefore $N \leq H$, where H is a Hall π-subgroup of G. But since H is abelian, $N = H$ is a Hall π-subgroup of G. Hence $l_\pi(G) \leq 1$.

A group G is called *π-closed* provided G has a normal Hall π-subgroup.

Lemma 3.5. *Let H be a Hall π-subgroup of G. If G is π-separable and $H \leq Z(N_G(H))$, then G is π'-closed.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. Then $G \neq H$. The class $\mathcal{F}$ of all π'-closed groups coincides with the product $\mathcal{G}_{\pi'}\mathcal{G}_\pi$. Hence $\mathcal{F}$ is a saturated formation by Lemma 3.2. Let N be a minimal normal subgroup of G. Since G is π-separable, N is a π-group or a π'-group. Moreover, G is a C_π-group by [34, 9.1.6]), so the hypothesis holds for

G/N. Hence G/N is π'-closed by the choice of G. Therefore N is the only minimal normal subgroup of G, $N \nleq \Phi(G)$ and N is a π-group. Therefore $N \leq H$ and $N = C_G(N)$ by [22, A, 15.2]. Since $H \leq Z(N_G(H))$ and H is a Hall π-subgroup of G, $N = H$. Therefore $N \leq Z(G)$, which implies that $N = H = G$. This contradiction completes the proof of the lemma.

4 Proof of Theorem A

Recall that G is a PST-group if and only if $G = D \rtimes M$, where $D = G^{\mathfrak{N}}$ is abelian Hall subgroup of G and every element $x \in M$ induces a power automorphism on D [3]. Therefore the implication (i) $\Rightarrow$ (ii) is a direct corollary of Theorem B.

Now suppose that $G = D \rtimes M$, where $D = G^{\mathfrak{N}}$, is a soluble PST-group. Let H be any subgroup of G and S a Hall π'-subgroup of H. Since G is soluble, we may assume without loss of generality that $S \leq M$. Hence $H = (D \cap H)(M \cap H) = (D \cap H)S$ and $D \cap H$ is normal in G. Let $\pi_1 = \pi(S)$. Let A be a Hall π_1-subgroup of M and E a complement to A in M. Then $E \leq C_G(S)$. Therefore $G = DM = DAE = N_G(H)(DA)$ and every subgroup L of DA satisfying $(|H|, |L|) = 1$ is contained in D. Thus H is quasipermutablein G. Thus (ii) $\Rightarrow$ (iii).

(iv) $\Rightarrow$ (ii) By Theorems C and D, G is supersoluble and D is a Hall subgroup of G. Therefore $G = D \rtimes W$, where W is a Hall π'-subgroup of G. By hypothesis, W is quasipermutable in G. Now arguing similarly as in the proof of Theorem B one can show that D is abelian and every subgroup of D is normal in G. Therefore G is a PST-group.

5 Final remarks

1. The subgroup S_3 is quasipermutable, S-propermutable and not propermutable in S_4. If H is the subgroup of order 3 in S_3, then H is S-quasipermutable and not quasipermutable in S_4.

2. Arguing similarly to the proof of Theorem A one can prove the following fact.

Theorem 5.1. *Suppose that G is soluble and let $\pi = \pi(G^{\mathfrak{N}})$. Then G is a PST-group if and only if every subnormal π-subgroup and a Hall π'-subgroup of G are propermutable in G.*

3. If G is metanilpotent, that is $G/F(G)$ is nilpotent, then for every Hall subgroup E of G we have $G = N_G(E)F(G)$. Therefore, in this case, every characteristic subgroup of every Hall subgroup of G is S-propermutable in G. In particular, every Hall subgroup of every supersoluble group is S-propermutable. This observation makes natural the following question: *What is the structure of G under the hypothesis that every Hall subgroup of G is propermutable in G ?* Theorem B gives an answer to this question.

4. Every maximal subgroup of a supersoluble group is quasipermutable. Therefore, in fact, Theorem A shows that the class of all soluble groups in which quaipermutability is a transitive relation coincides with the class of all soluble PST-groups.

5. We say that G is a *SQT-group* if S-quasipermutability is a transitive relation in G. Arguing similarly to the proof of Theorem A one can prove the following fact.

Theorem 5.2. *A soluble group G is an SQT-group if and only if $G = D \rtimes M$ is supersoluble, where D and M are Hall nilpotent subgroups of G and the index $|G : DN_G(H \cap D)|$ is a $\pi(H)$-number for every subgroup H of G.*

6. A subgroup H of G is called *SS-quasinormal* [21] (*semi-normal* [33]) in G provided G has a subgroup B such that $HB = G$ and H permutes with all Sylow subgroups (H permutes with all subgroups, respectively) of B.

It is clear that every SS-quasinormal subgroup is S-propermutable and every semi-normal subgroup is propermutable. Moreover, there are simple examples (consider, for example, the group $C_7 \rtimes \mathrm{Aut}(C_7)$, where C_7 is a group of order 7) which show that, in general, the class of all S-propermutable subgroups of G is wider than the class of all its SS-quasinormal subgroups and the class of all propermutable subgroups of G is wider than the class of all its semi-normal subgroups. Therefore Proposition covers main results (Theorems 1.1–1.5) in [21].

7. Theorem 3.1 is used in the proof of Theorem B. From this result we also get

Corollary 5.3 (See [35, Theorem 5.4]). *Let H be a Hall semi-normal subgroup of G. If $p > q$ for all primes p and q such that p divides $|H|$ and q divides $|G : H|$, then H is normal in G.*

Corollary 5.4 (See [36, Theorem]). *Let P be a Sylow p-subgroup of G. If P is semi-normal in G, then the following statements hold:*

(i) *G is p-soluble and $P' \leq O_p(G)$.*

(ii) $l_p(G) \leq 2$.

(iiii *If for some prime $q \in \pi'$ a Hall p'-subgroup of G is q-supersoluble, then G is q-supersoluble.*

Corollary 5.5 (See [37, Theorem 3]). *If a Sylow p-subgroup P of G, where p is the largest prime dividing $|G|$, is semi-normal in G, then P is normal in G.*

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